International Journal of Theoretical Physics, Vol. 43, No. 6, June 2004 (© 2004)

New Method of Generating Exact Inflationary Solutions in Generalized Einstein Theories

Wenfu Wang[1] and Shuzheng Yang[2,3]

Received

We discuss the general approach to finding exact inflationary solutions in generalized Einstein theories. These solutions are found by taking the Hubble parameter directly as a function of the field φ and then determining the evolution of the expansion scale factor and the potential from it. This method allows the full dynamic behavior of the field to be investigated in terms of the function $H(\varphi)$ without needing to assume that friction terms in the field equations dominate or that the field's kinetic energy is negligible.

KEY WORDS: Einstein theory; exact inflationary solution; scale factor.

1. INTRODUCTION

The inflationary scenario seeks to solve some puzzling cosmological questions (like the flatness and the isotropy problems) that are present in the standard Big Bang theory, and it provides a mechanism for the generation of density perturbations needed to seed the formation of structures in the universe (Guth, 1981; Linde, 1982, 1990). The essential qualitative feature of inflation, the acceleration of the universe, is also required (albeit at a different rate) in the present epoch of the universe in order to explain the data from high red shift supernovae (Perlmutter *et al.*, 1998; Riess *et al.*, 1998).

The action describing the interaction between the gravitational field and a scalar field is generally constructed using the minimal coupling principle (i.e. $\xi = 0$). Recently, generalized versions of gravity theories in reconstructing the early universe scenario have attracted a great deal of attention as a possible improvement of the minimal coupling case (Fakir and Unruh, 1992; Faraoni, 1996).

[1] Department of Industrial Automation, Southwest Petroleum Institute, Nanchong, Sichuan, People's Republic of China.
[2] College of Physics and Electric Information, China West Normal University, Nanchong, Sichuan, People's Republic of China.
[3] To whom correspondence should be addressed at College of Physics and Electric Information, China West Normal University, Nanchong, Sichuan 637002, People's Republic of China; e-mail: szyang_sc@sohu.com.

There are many compelling reasons to include an explicit nonminimal (i.e. $\xi \neq 0$) coupling in the action. In many models of the very early Universe, the canonical Einstein–Hilbert gravitational action emerges only as a low-energy effective theory, rather than being assumed from the start (Adler, 1982). Naturally, assuming that the early universe was dominated by some of these gravity theories, many inflation models have been proposed based on generalized gravity theories. These other-than-Einstein gravity theories naturally arise either from attempts to quantify gravity (Birrell and Davies, 1982) or as the low-energy limits of the unified theories including gravity (Green *et al.*, 1987). In principle, the consideration of such an interaction allows one to take more properly into account the influence that, in the very early Universe, the extremely high value of the curvature potentially had in the dynamic behavior of this coupled system.

It is common to think that the inflationary scenario could be driven by the presence of a scalar field with inflation potential. Most detailed studies of inflation have been made by using numerical integration, or by employing an approximation scheme. The "slow-roll approximation" (Albrecht and Stenhardt, 1982; Guth, 1981; Linde, 1982), which neglects the most slowly changing terms in the equations of motion, is the one used most widely. Although this approximation works well in many cases, we know that it must eventually fail if inflation has to end. Moreover, even weak violations of it can result in significant deviations from the standard predications for observables such as the spectrum of density perturbations or the density of gravitational waves in the Universe. It is more difficult to achieve the slow rolling of the scalar field when $\xi \neq 0$. In fact, an almost flat section of the potential $V(\varphi)$ gives a slow rollover of φ when $\xi = 0$, but its shape is distorted by the nonminimal coupling term $\xi R\varphi^2/2$ in the Lagrangian density. The extra term plays the role of an effective mass term for the inflation (Futamase and Maeda, 1989).

Slow roll is not, however, the only possibility for successfully implementing models of inflation, and solutions outside the slow-roll approximation have been found in minimal coupling cases (Wang, 2001a,b).

In this paper, we will extend our previous results adding to an explicit nonminimal coupling in the action. We will discuss a general approach to finding exact inflationary solutions in generalized Einstein theories. Exact inflationary solutions are best achieved by expressing the Hubble parameter H during inflation as a function of the scalar field φ. This involves using the inflation as an effective time coordinate and allows the full dynamic behavior of the field to be investigated in terms of the function $H(\varphi)$ without needing to assume that friction terms in the field equations dominate or that the field's kinetic energy is negligible (Carr and Lidsey, 1993).

This paper is organized as follows. In Section 2, we give a new method of generating exact inflationary solutions in generalized Einstein theories. In Section 3, exact inflationary solutions in induced-gravity theory are discussed. In Section

4, exact inflationary solutions with nonminimal coupling are computed. Section 5 contains the conclusions.

2. A NEW METHOD OF GENERATING EXACT INFLATIONARY SOLUTIONS IN GENERALIZED EINSTEIN THEORIES

In inflationary theory it is assumed that the scalar field dominates the evolution of the universe and that no forms of matter other than the scalar field φ are included in the Lagrangian density. We shall adopt in the present paper the sign convention for ξ such that the conformal coupling means $\xi = -1/6$ (Fakir and Unruh, 1990). We start our analysis by searching for exact cosmological solutions from a generic action where a scalar field φ is no minimally coupled with gravity

$$I = \int d^4x\sqrt{-g}\left[F(\varphi)R + \frac{1}{2}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - V(\varphi)\right], \tag{1}$$

where $V(\varphi)$ is generic potential for the scalar field φ and $F(\varphi)$the coupling for the field φ. In this paper we restrict ourselves to

$$\text{Induced gravity } F(\varphi) = \frac{1}{2}\xi\varphi^2, \tag{2}$$

$$\text{no minimally coupling } F(\varphi) = \frac{1}{16\pi G} + \frac{1}{2}\xi\varphi^2. \tag{3}$$

We assume a homogeneous distribution for the scalar field, and the line element is taken to be that of the Robertson–Walker universe

$$ds^2 = dt^2 - a^2(t)\,dx^2, \tag{4}$$

where $a(t)$ is the scalar factor and the spatial curvature, which is unimportant in this context, is set to zero.

Variation of the action with respect to the gravitational degrees of freedom yields Einstein's equations,

$$F(\varphi)G_{\mu\nu} = -\frac{1}{2}\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{;a}\varphi^{;a} + g_{\mu\nu}V(\varphi) + g_{\mu\nu}F(\varphi) - \quad \Box\, F(\varphi)_{;\mu;\nu} \tag{5}$$

where $G_{\mu\nu}$is the Einstein tensor and

$$\Box\, F(\varphi) = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta F(\varphi)). \tag{6}$$

Taking the time–time component of Eq. (5) gives the energy equation

$$3H^2 = \frac{1}{2F(\varphi)}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) - 6F'(\varphi)H\dot{\varphi}\right], \tag{7}$$

where over dots denote time derivatives, $H = \dot{a}/a$ is the Hubble expansion rate and $F'(\varphi) \equiv dF(\varphi)/d\varphi$.

Variation with respect to the matter fields results in the field equation

$$\ddot{\varphi} + 3H\dot{\varphi} - 6F'(\varphi)(\dot{H} + 2H^2) + V'(\varphi) = 0, \tag{8}$$

where $V'(\varphi) \equiv dV(\varphi)/d\varphi$.

Finally, combining the time derivative of Eq. (7) with Eq. (8) yields the momentum equation,

$$\dot{H} = \frac{1}{2F(\varphi)}\left[-\frac{1}{2}\dot{\varphi}^2 + F'(\varphi)\left(H\dot{\varphi} - \frac{\dot{\varphi}^2}{\varphi} - \ddot{\varphi}\right)\right]. \tag{9}$$

Clearly, according to the Bianchi identity, there are only two independent field equations.

From Eqs. (7)–(9), we find

$$\dot{\varphi} = \frac{-(6F\varphi + 12F'^2\varphi + 12FF')H^2 + (\varphi + 2F')V + \varphi F'V'}{(2F\varphi + 6\varphi F'^2)H' + (2\varphi F' + 12F'^2)H}, \tag{10}$$

where $H'(\varphi) \equiv dH/d\varphi$. Using Eq. (7) we have

$$\dot{\varphi} = 6F'H \pm \Phi(\varphi), \tag{11}$$

where

$$\Phi(\varphi) \equiv [12(3F'^2 + F)H^2 - 2V]^{1/2}. \tag{12}$$

It is shown that for$F(\varphi) = 1\big/16\pi G$, using Eqs. (7)–(12) we recover the field equation in the minimal coupling case (Wang, 2001c)

$$\dot{\varphi} = -\frac{H'}{4\pi G}. \tag{13}$$

From Eq. (13) one obtains that in Eq. (11) the positive sign corresponds to $H'(\varphi) < 0$ whereas the negative sign corresponds to $H'(\varphi) > 0$.

Equation (12) can be rewritten as

$$V(\varphi) = 18F'^2H^2 + 6FH^2 - \frac{1}{2}\Phi^2(\varphi). \tag{14}$$

Using Eqs. (10), (11), and (14), we obtain

$$\Phi'(\varphi) + \left(\frac{1}{2F'} + \frac{1}{\varphi}\right)\Phi = \mp\left[\left(\frac{2F}{F'} + 6F'\right)H' + \left(2 + \frac{12F'}{\varphi}\right)H\right]. \tag{15}$$

Its general solution is

$$\Phi(\varphi) = C(\varphi)\varphi^{-(k+1)}, \tag{16}$$

where $k \equiv 1\big/(2\xi)$ and the function $C(\varphi)$ is determined by

$$C'(\varphi) = \mp\varphi^{k+1}\left[\left(\frac{2F}{F'} + 6F'\right)H' + \left(2 + \frac{12F'}{\varphi}\right)H\right], \tag{17}$$

where a prime denotes differentiation with respect to φ.

We now have all the ingredients for a recipe to obtain exact inflationary solutions. The steps are 1) from$H(\varphi)$, calculate $\Phi(\varphi)$ using Eqs. (16) and (17), 2) calculate $V(\varphi)$ using Eq. (14), 3) calculate $\varphi(t)$ from Eq. (11), 4) from $H(\varphi)$ and $\varphi(t)$ calculate $a(t)$. For this reason, it has been suggested that it is more efficient to begin by specifying the form of $H(\varphi)$, rather than $V(\varphi)$.

3. EXACT INFLATIONARY SOLUTIONS IN INDUCED-GRAVITY THEORY

The action, Eq. (1), can be used to study induced gravity. In this section we consider$F(\varphi) = \xi\varphi^2/2$.

From Eq. (11) one obtains the following equation to be satisfied by $\varphi(t)$,

$$\dot{\varphi} = \frac{3}{k}\varphi H \pm \Phi(\varphi), \tag{18}$$

The function $C(\varphi)$ is determined by

$$C'(\varphi) = \mp\varphi^{k+1}\left[\left(1 + \frac{3}{k}\right)\varphi H' + 2\left(1 + \frac{3}{k}\right)H\right]. \tag{19}$$

A number of particular cases are instructive:

Example 3.1. The simplest case is

$$H(\varphi) = H_0 \equiv \text{const.} \tag{20}$$

From Eqs. (19) and (20) we find

$$C(\varphi) = 2\left(1 + \frac{3}{k}\right)\frac{H_0}{k+2}\varphi^{k+2}, \tag{21}$$

where the integration constant is assumed to be zero. From Eqs. (16) and (21) we have

$$\Phi(\varphi) = \frac{2(k+3)}{k(k+2)}H_0\varphi. \tag{22}$$

An exact solution of Eq. (18) is

$$\varphi = \varphi_0 \exp\left(\frac{H_0}{k+2}t\right), \tag{23}$$

where φ_0 is the value of $\varphi(t)$ at the beginning of the inflationary epoch ($t = 0$). Using Eqs. (14), (20), and (22) the potential driving this evolution is given as

$$V(\varphi) = \frac{1}{2}m^2\varphi^2, \tag{24}$$

where

$$m^2 \equiv \frac{(k+3)(3k+8)}{k(k+2)^2}H_0^2. \tag{25}$$

From Eq. (20) we find

$$a = a_0 e^{H_0 t}, \tag{26}$$

where a_0 is the value of $a(t)$ at the beginning of the inflationary epoch ($t = 0$). Thus Eqs. (20) and (23)–(26) give the complete exact inflationary solution.

Example 3.2. The ansatz for the exact solution is

$$H(\varphi) = \frac{\alpha}{\varphi}, \tag{27}$$

where α is positive constant parameter.

From Eqs. (19) and (27) we find

$$C(\varphi) = -\frac{k+3}{k(k+1)}\alpha\varphi^{k+1}, \tag{28}$$

where integration constant is assumed to be zero. From Eqs. (16) and (28) we have

$$\Phi(\varphi) = -\frac{k+3}{k(k+1)}\alpha. \tag{29}$$

An exact solution of Eq. (18) is

$$\varphi = \varphi_0 + \frac{2\alpha}{k+1}t, \tag{30}$$

where φ_0 is the value of $\varphi(t)$ at the beginning of the inflationary epoch ($t = 0$). Using Eqs. (14), (27), and (29) the potential driving this evolution is given as

$$V(\varphi) = \frac{(k+3)(3k+5)}{2k(k+1)^2}. \tag{31}$$

From Eqs. (27) and (30) we find

$$a = a_0\left(1 + \frac{2\alpha}{(k+1)\varphi_0}t\right)^{(k+1)/2\alpha}, \tag{32}$$

where a_0 is the value of $a(t)$ at the beginning of the inflationary epoch ($t = 0$). Thus when $(k+1)/2\alpha > 1$, Eqs. (27) and (30)–(32) give the complete exact inflationary solution. It is worthwhile to point here that, for the case of the minimal coupling power-law inflation is normally driven by exponential potentials $V(\varphi) \propto \exp(-\lambda\varphi)$, with λ constant >0. From Eq. (32), in the induced-gravity case power-law inflation is driven by a constant potential.

Example 3.3. Let us assume

$$H(\varphi) = \alpha - \varphi^2, \tag{33}$$

where α is a positive constant parameter.

From Eqs. (19) and (33) we find

$$C(\varphi) = 2\left(1 + \frac{3}{k}\right)\left[\frac{\alpha}{k+2}\varphi^{k+2} - \frac{2}{k+4}\varphi^{k+4}\right], \tag{34}$$

where integration constant is assumed to be zero. From Eqs. (16) and (34) we have

$$\Phi(\varphi) = 2\left(1 + \frac{3}{k}\right)\left[-\frac{\alpha}{k+2}\varphi + \frac{2}{k+4}\varphi^3\right]. \tag{35}$$

An exact solution of Eq. (18) is

$$\frac{\varphi}{\sqrt{A + B\varphi^2}} = \frac{\varphi_0}{\sqrt{A + B\varphi_0^2}} e^{At}, \tag{36}$$

where $A \equiv \alpha/(k+2)$ and $B \equiv 1/(k+4)$ are positive parameters; φ_0 is the value of φ at the beginning of the inflationary epoch ($t = 0$). The potential driving of this evolution is given from Eqs. (14), (33), and (35) as

$$V(\varphi) = A_1\varphi^2 + A_2\varphi^4 + A_3\varphi^6, \tag{37}$$

where

$$A_1 \equiv \frac{(k+3)(3k+8)}{2k(k+2)^2}\alpha^2, \tag{38}$$

$$A_2 \equiv -\frac{(k+3)(3k+10)}{k(k+2)(k+4)}\alpha, \tag{39}$$

$$A_3 \equiv \frac{(k+3)(3k+8)}{2k(k+4)^2}. \tag{40}$$

From Eqs. (33) and (36) we find

$$a = a_0 e^{\alpha t}\left[1 + \frac{B\varphi_0^2(1 - e^{2At})}{A}\right]^{1/2A}, \tag{41}$$

where $a_0$is the value of $a(t)$at the beginning of the inflationary epoch ($t = 0$). Thus when $\xi < 1/2$, Eqs. (33) and (36)–(41) is a possible inflationary model.

4. EXACT INFLATIONARY SOLUTIONS WITH NONMINIMAL COUPLING

The action, (1), can be used to study a nonminimal coupling similar to, but distinct from, that of induced-gravity inflation. In this section we consider $F(\varphi) = 1/16\pi G + \xi\varphi^2/2$. In this case Eq. (11) can be rewritten as

$$\dot{\varphi} = \frac{3}{k}\varphi H \pm \Phi(\varphi), \tag{42}$$

where

$$\Phi(\varphi) \equiv \left[\left[\frac{3}{4\pi G} + \frac{3}{k}\left(1 + \frac{3}{k}\right)\varphi^2\right]H^2 - 2V\right]^{1/2}. \tag{43}$$

Equation (15) can be rewritten as

$$\Phi'(\varphi) + \frac{k+1}{\varphi}\Phi = \mp\left[\left[\frac{k}{4\pi G\varphi} + \left(1 + \frac{3}{k}\right)\varphi\right]H' + 2\left(1 + \frac{3}{k}\right)H\right]. \tag{44}$$

The function $C(\varphi)$ is determined by

$$C'(\varphi) = \mp\varphi^{k+1}\left[\left(\frac{k}{4\pi G\varphi} + \left(1 + \frac{3}{k}\right)\varphi\right)H' + 2\left(1 + \frac{3}{k}\right)H\right]. \tag{45}$$

A number of particular cases are instructive:

Example 4.4. The simplest case is

$$H(\varphi) = H_0 \equiv \text{const.} \tag{46}$$

From Eq. (45) we find

$$a = a_0 e^{H_0 t}, \tag{47}$$

where a_0 is the value of $a(t)$ at the beginning of the inflationary epoch ($t = 0$). From Eq. (45) we find

$$C(\varphi) = -\frac{2(k+3)}{k(k+2)}H_0\varphi^{k+2} + C_1, \tag{48}$$

where C_1 is integration constant. From Eq. (16) we find

$$\Phi(\varphi) = -\frac{2(k+3)}{k(k+2)}H_0\varphi + C_1\varphi^{-(k+1)}. \tag{49}$$

Using Eqs. (43), (46), and (49) we find

$$V(\varphi) = \frac{3H_0^2}{8\pi G} + \frac{(k+3)(3k+8)}{2k(k+2)^2}H_0^2\varphi^2 + \frac{2(k+3)}{k(k+2)}C_1 H_0\varphi^{-k} - \frac{1}{2}C_1^2\varphi^{-2(k+1)}. \tag{50}$$

In particular, for the conformally coupling ($k = -3$), then Eq. (50) implies

$$V(\varphi) = \frac{3H_0^2}{8\pi G} - \frac{1}{2}C_1^2\varphi^4. \tag{51}$$

If we take that $V_0 \equiv 3H_0^2/8\pi G$ and $2C_1^2 \equiv \lambda$, Eq. (51) is rewritten as

$$V(\varphi) = V_0 - \frac{\lambda}{4}\varphi^4. \tag{52}$$

An exact solution of Eq. (42) is

$$\varphi = \frac{H_0}{\alpha - ce^{H_0 t}}, \tag{53}$$

where $\alpha \equiv C_1$, and $c = \alpha - H_0/\varphi_0$, in which φ_0 is the value of $\varphi(t)$ at $t = 0$. The result is the same as that shown by Wang (1997).

Example 4.5. Let us take the case

$$H(\varphi) = \alpha_1\varphi^2 + \alpha_2, \tag{54}$$

where $\alpha_1$and α_2 are positive constant parameters. From Eqs. (45) and (54) we find

$$C'(\varphi) = \left[\frac{k\alpha_1}{2\pi G} + 2\left(1 + \frac{3}{k}\right)\alpha_2\right]\varphi^{k+1} + 4\left(1 + \frac{3}{k}\right)\alpha_1\varphi^{k+3}]. \tag{55}$$

Its general solution is

$$C(\varphi) = A\varphi^{k+2} + B\varphi^{k+4}, \tag{56}$$

where

$$A \equiv \frac{\alpha_1 k}{2\pi G(k+2)} + \frac{2\alpha_2(k+3)}{k(k+2)}, \tag{57}$$

$$B \equiv \frac{4\alpha_1(k+3)}{k(k+4)}. \tag{58}$$

From Eqs. (16) and (56) we have

$$\Phi = A\varphi + B\varphi^3. \tag{59}$$

An exact solution of Eq. (42) is

$$\varphi(t) = \left[\frac{A - \frac{3\alpha_2}{k}}{C_1 \exp\left[-2\left(\frac{3\alpha_2}{k} - A\right)t\right] - \left(B - \frac{3\alpha_1}{k}\right)}\right]^{1/2}, \tag{60}$$

where C_1 is integration constant. The scalar field potential is given then by:

$$V(\varphi) = \left[\frac{3}{2k}\left(1 + \frac{3}{k}\right)\alpha_1^2 - \frac{1}{2}B^2\right]\varphi^6 + \left[\frac{3}{k}\left(1 + \frac{3}{k}\right)\alpha_1\alpha_2 + \frac{3\alpha_1^2}{8\pi G} - AB\right]\varphi^4$$

$$+\left[\frac{3}{2k}\left(1+\frac{3}{k}\right)\alpha_2^2+\frac{3\alpha_1\alpha_2}{4\pi G}-\frac{1}{2}A^2\right]\varphi^2+\frac{3\alpha_2^2}{8\pi G}. \tag{61}$$

From Eqs. (54) and (60) we find

$$a = C_2\left[C_1\exp\left[-2\left(\frac{3\alpha_2}{k}-A\right)t\right]\right.$$
$$\left.-\left(B-\frac{3\alpha_1}{k}\right)\right]^{\alpha_1/2(B-3\alpha_1/k)}\exp\left[\left(\alpha_2-\frac{\alpha_1(Ak-3\alpha_2)}{Bk-3\alpha_1}\right)t\right], \tag{62}$$

where C_2 is integration constant. Thus Eqs. (54) and (60)–(62) give the complete exact inflationary solution.

5. CONCLUSIONS

We discuss the general approach to finding exact inflationary solutions in the generalized Einstein theories. These solutions are found by taking the Hubble parameter directly as a function of the field φ and then determining the evolution of the expansion scale factor and the potential from it. This allows the full dynamical behavior of the field to be investigated in terms of the function $H(\varphi)$ without needing to assume that friction terms in the field equations dominate or that the field's kinetic energy is negligible. Moreover, since in principle one may express all the relevant dynamical information about scalar field models via the function $H(\varphi)$ and its first derivative, this function may serve as an interesting dynamical variable to study the models analytically. Therefore, it becomes relatively easy to construct exact solutions for different functions of $H(\varphi)$.

We have presented simple procedure to construct exact solutions for scalar field isotropic and spatially flat cosmologies with potential of different shapes. It is hoped that this method will become fruitful for generating and studying models of physical interest.

REFERENCES

Adler, S. L. (1982). *Reviews of Modern Physics* **54**, 729.
Albrecht, A. and Stenhardt, P. J. (1982). *Physical Review Letters* **48**, 1220.
Birrell, N. D. and Davies, P. C. (1982). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, England.
Carr, B. J. and Lidsey, J. E. (1993). *Physical Review D* **48**, 543.
Fakir, R. and Unruh, W. G. (1990). *Physical Review Letters* **41**, 1783.
Fakir, R. and Unruh, W. G. (1992). *Astrophysical Journal* **394**, 396.
Faraoni, V. (1996). *Physical Review D* **53**, 6813.
Futamase, T. and Maeda, K. (1989). *Physical Review D* **39**, 399.
Green, M., Schwarz, J., and Witten, E. (1987). *Superstring Theory*, Cambridge University Press, Cambridge, England.

Guth, A. H. (1981). *Physical Review D* **23**, 347.
Linde, A. D. (1982). *Physics Letters B* **108**, 389.
Linde, A. D. (1990). *Particle Physics and Inflationary Cosmology*, Hardwood Academic, Chur, Switzerland.
Perlmutter, S., *et al*. (1998). *Nature* (*London*) **391**, 51.
Riess, A. G., *et al*. (1998). *Astrophysical Journal* **116**, 1009.
Wang, W. F. (1997). *Physics Letters A* **237**, 10.
Wang, W. F. (2001a). *Chinese Physics Letters* **18**, 997.
Wang, W. F. (2001b). *Communications in Theoretical Physics* (*Beijing, China*) **36**, 509.
Wang, W. F. (2001c) *Chinese Physics Letters* **18**, 151.